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Gauge and BRST generators for space-time non-commutative U(1) theory

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Abstract: The hamiltonian (gauge) symmetry generators of non-local (gauge) theories are presented. The construction is based on the $d+1$ dimensional space-time formulation of d dimensional non-local theories. The procedure is applied to U(1) space-time non-commutative gauge theory. In the hamiltonian formalism the hamiltonian and the gauge generator are constructed. The nilpotent BRST charge is also obtained. The Seiberg-Witten map between non-commutative and commutative theories is described by a canonical transformation in the superphase space and in the field-antifield space. The solutions of classical master equations for non-commutative and commutative theories are related by a canonical transformation in the antibracket sense.

Keywords: BRST Quantization, Brane Dynamics in Gauge Theories, Gauge Symmetry, Non-Commutative Geometry.

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Contents

1. Introduction

Non-local theories are described by actions that contain an infinite number of temporal derivatives. There exists an equivalent formulation of those theories in a space-time of one dimension higher [1]. In this formulation there are two time coordinates, and the dynamics in this space is described in such a way that the evolution is local with respect to one of the times. Thanks to this, a hamiltonian formalism can be constructed in the $d+1$ dimensions as a local theory with respect to the evolution time [1]–[4], in which the Euler-Lagrange equations appear as hamiltonian constraints [2]. A characteristic feature is that there is no dynamics in the usual sense; i.e. the physical trajectories are not obtained as evolution of some given initial conditions.

In this paper we construct the symmetry generators for non-local theories. Corresponding to symmetries of a non-local lagrangian the symmetry generators are constructed in a natural way in $d+1$ dimensions and are conserved quantities. When the original symmetries of the non-local theory are gauge symmetries the corresponding transformations are realized as rigid symmetries in the $d+1$ dimensions.

We analyze in detail the case of space-time non-commutative (NC) U(1) gauge theory.[1] In particular, we obtain its hamiltonian and we show that it is the generator of time translations. We also study the relation between the gauge generators of the NC and commutative theories, by considering the Seiberg-Witten (SW) map [5] as an ordinary canonical transformation.

We then move to study the BRST symmetry of this U(1) NC theory and we construct the nilpotent hamiltonian BRST charges. We also analyze the BRST symmetry at lagrangian level using the field-antifield formalism. The Seiberg-Witten (SW) map [5] is extended to a canonical transformation in superphase space and in the field-antifield space. We show that the solutions of the classical master equation for non-commutative and commutative theories are related by a canonical transformation in the antibracket sense.

The organization of the paper is as follows. In section 2 we study the general properties of symmetry generators of non-local theories. In section 3 we construct the gauge symmetry generator for U(1) NC gauge theory. Section 4 is devoted to study the relation between the gauge generators of commutative and U(1) NC gauge theories. In section 5 we construct the BRST generator. There is an appendix A where the ordinary U(1) local Maxwell theory is analyzed in terms of the $d+1$ dimensional formalism.

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2. Hamiltonian formalism of non-local theories and symmetry generators

2.1 Brief review

A non-local lagrangian at time t depends not only on variables at time t but also on ones at different times. In other words it depends on an infinite number of time derivatives of the positions $q_i(t)$.[2] The analogue of the tangent bundle for lagrangians depending on positions and velocities is now infinite dimensional. It is the space of all possible trajectories. The action is

$$S[q] = \int dt\ L^{\rm non}(t)\,. \tag{2.1}$$

[1]Here we use the term "U(1)" for "rank one" gauge field. It is not abelian for the NC case.

[2]For simplicity, in this section we will explicitly consider the case of mechanics.

The Euler-Lagrange (EL) equation is obtained by taking the functional variation of (2.1),

$$\frac{\delta S[q]}{\delta q_i(t)} = \int dt' E^i(t', t; [q]) = 0\,, \qquad E^i(t', t; [q]) \equiv \frac{\delta L^{\rm non}(t')}{\delta q_i(t)}\,. \tag{2.2}$$

One of the most striking features of such theories is that of the new interpretation that this EL equation has [1, 2]. Since the equations of motion are of infinite degree in time derivatives, one should give as initial conditions the value of all these derivatives at some initial time. In other words, we should give the whole trajectory (or part of it) as the initial condition. If we denote the space of all possible trajectories as $J = \{q(\lambda), \lambda \in R\}$, then (2.2) is a lagrangian constraint defining the subspace $J_R \subset J$ of physical trajectories.

In [1, 2] this was implemented using a formalism in which one works with one extra dimension. The final result was that one obtains a two dimensional field theory whose lagrangian is

$$\tilde{L}(t, [\mathcal{Q}]) := \int d\sigma \delta(\sigma) \mathcal{L}(t, \sigma)\,, \tag{2.3}$$

where the lagrangian density $\mathcal{L}(t, \sigma)$ is constructed from the original non-local one $L^{\rm non}$ by performing the following replacements

$$q_i(t) \to \mathcal{Q}_i(t, \sigma)\,, \qquad \frac{d^n}{dt^n} q_i(t) \to \frac{\partial^n}{\partial \sigma^n} \mathcal{Q}_i(t, \sigma)\,, \qquad q_i(t + \rho) \to \mathcal{Q}_i(t, \sigma + \rho)\,. \tag{2.4}$$

Note that this $1+1$ field theory has two "time" coordinates t and σ but, using these replacements, the dynamics is described in such a way that the evolution is local with respect to one of them (t). This is the key achievement that will enable us to analyze many aspects of the $1+1$ theory using ordinary methods from local theories.

The theory was also shown to have the following hamiltonian

$$H(t) = \int d\sigma [\mathcal{P}^i(t, \sigma) \mathcal{Q}_i{}'(t, \sigma) - \delta(\sigma) \mathcal{L}(t, \sigma)]\,, \tag{2.5}$$

where $\mathcal{Q}'_i(t, \sigma) \equiv \partial_\sigma \mathcal{Q}_i(t, \sigma)$ and $\mathcal{P}^i(t, \sigma)$ are the canonical momenta. Note that the hamiltonian depends on the fields $\mathcal{Q}_i(t, \sigma)$ and an infinite number of sigma derivatives of it, but not on any derivative with respect to the evolution time t. Thus the hamiltonian (2.5) is indeed a well defined phase space quantity.

The Hamilton equations are, denoting time (t) derivatives by "dots",

$$\dot{\mathcal{Q}}_i(t, \sigma) = \mathcal{Q}'_i(t, \sigma)\,, \tag{2.6}$$

$$\dot{\mathcal{P}}^i(t, \sigma) = \mathcal{P}^{i\prime}(t, \sigma) + \frac{\delta \mathcal{L}(t, 0)}{\delta \mathcal{Q}_i(t, \sigma)} = \mathcal{P}^{i\prime}(t, \sigma) + \mathcal{E}^i(t; 0, \sigma)\,, \tag{2.7}$$

where $\mathcal{E}(t;\sigma',\sigma)$ is defined by

$$\mathcal{E}^i(t;\sigma',\sigma) = \frac{\delta \mathcal{L}(t,\sigma')}{\delta \mathcal{Q}_i(t,\sigma)} \,. \tag{2.8}$$

Equations (2.6) restrict the two dimensional fields $\mathcal{Q}_i(t,\sigma)$ to depend only on a chiral combination of the two times $t+\sigma$ on shell. They are identified with the position variables $q_i(t)$ of the original system by

$$\mathcal{Q}_i(t,\sigma) = q_i(t+\sigma)\,, \qquad \text{i.e.} \qquad q_i(\sigma) \;=\; \mathcal{Q}_i(0,\sigma)\,. \tag{2.9}$$

The solutions to these 1+1 dimensional field equations are related to those of the EL equations (2.2) of the original non-local lagrangian $L^{\rm non}$ if we impose a constraint on the momentum [1]

$$\varphi^i(t,\sigma) = \mathcal{P}^i(t,\sigma) - \int d\sigma' \chi(\sigma,-\sigma')\mathcal{E}^i(t;\sigma',\sigma) \approx 0\,, \tag{2.10}$$

where $\chi(\sigma,-\sigma')$ is defined by using the sign distribution $\epsilon(\sigma)$ as $\chi(\sigma,-\sigma') = (\epsilon(\sigma)-\epsilon(\sigma'))/2$. We use *weak equality* symbol "$\approx$" for equations those hold on the constraint surface [6]. As usual, one has to impose stability to this constraint, leading us to the following one

$$\dot{\varphi}^i(t,\sigma) \approx \delta(\sigma)\ [\int d\sigma'\ \mathcal{E}^i(t;\sigma',0)] \approx 0\,. \tag{2.11}$$

We should require further consistency conditions of this constraint. Repeating this we get an infinite set of hamiltonian constraints which can be expressed collectively as

$$\tilde{\varphi}^i(t,\sigma) = \int d\sigma' \mathcal{E}^i(t;\sigma',\sigma) \approx 0\,, \quad (-\infty < \sigma < \infty)\,. \tag{2.12}$$

If we use (2.6) and (2.9) it reduces to the EL equation (2.2) of $q_i(t)$ obtained from $L^{\rm non}(t)$. This is precisely what we were seeking at the beginning, since now we see that the new 1+1 hamiltonian system incorporates the EL equation as a constraint on the phase space.

Summarizing, we have been able to describe the original non-local lagrangian system as a $1+1$ dimensional local (in one of the times) hamiltonian system, governed by the hamiltonian (2.5) and the constraints (2.10) and (2.12). The formalism introduced here can be thought of as a generalization of the Ostrogradski formalism [7] to the case of infinite derivative theories.

2.2 Hamiltonian symmetry generators

For local theories symmetry properties of the system are examined using the Nöether theorem [8]. In hamiltonian formalism the relation between symmetries and conservation laws has been discussed extensively for singular lagrangian systems, for example [9]–[11]. In this section, we develop a formalism to treat the case of non-local theories.

Suppose we have a non-local lagrangian , (2.1), which is invariant under some transformation $\delta q(t)$ up to a total derivative,

$$\delta L^{\rm non}(t) = \int dt' \, \frac{\delta L^{\rm non}(t)}{\delta q_i(t')} \, \delta q_i(t') = \frac{d}{dt} k(t) \, . \tag{2.13}$$

Now we move to our $1+1$ dimensional theory and take profit of the fact that it was local in the evolution time t. Therefore, we can construct the corresponding symmetry generator in the hamiltonian formalism in the usual way

$$G(t) = \int d\sigma \, \left[\mathcal{P}^i(t,\sigma) \delta \mathcal{Q}_i(t,\sigma) - \delta(\sigma) \mathcal{K}(t,\sigma) \right] , \tag{2.14}$$

where $\delta\mathcal{Q}_i(t,\sigma)$ and $\mathcal{K}(t,\sigma)$ are constructed from $\delta q(t)$ and $k(t)$ respectively by the same replacement (2.4), as $\mathcal{L}(t,\sigma)$ was obtained from $L^{\rm non}(t)$. The quasi-invariance of the non-local lagrangian (2.13), translated to the $1+1$ language, means

$$\int d\sigma' \frac{\delta \mathcal{L}(t,\sigma)}{\delta \mathcal{Q}_i(t,\sigma')} \delta \mathcal{Q}_i(t,\sigma') = \partial_\sigma \mathcal{K}(t,\sigma) \, . \tag{2.15}$$

When the original non-local lagrangian has a gauge symmetry the $\delta q_i(t)$ and $k(t)$ contain an arbitrary function of time $\lambda(t)$ and its t derivatives. In $\delta\mathcal{Q}_i(t,\sigma)$ and $\mathcal{K}(t,\sigma)$ the $\lambda(t)$ is replaced by $\Lambda(t,\sigma)$ in the same manner as $q_i(t)$ is replaced by $\mathcal{Q}_i(t,\sigma)$ in (2.4). However in order for the transformation generated by (2.14) to be a symmetry of the Hamilton equations, $\Lambda(t,\sigma)$ can not be an arbitrary function of t but should satisfy

$$\dot\Lambda(t,\sigma) = \Lambda'(t,\sigma) \tag{2.16}$$

as will be shown shortly. This restriction on the parameter function Λ means that the transformations generated by $G(t)$ in the $d+1$ dimensional hamiltonian formalism are rigid transformations in contrast with the original ones for the non-local theory which are gauge transformations. In the appendix we will see how this rigid transformations in the $d+1$ dimensional hamiltonian formalism are reduced to the usual gauge transformations in d dimension for the U(1) Maxwell theory.

The generator $G(t)$ generates the transformation of $\mathcal{Q}_i(t,\sigma)$,

$$\delta\mathcal{Q}_i(t,\sigma) = \{ \mathcal{Q}_i(t,\sigma), \; G(t) \} \, , \tag{2.17}$$

corresponding to the transformation $\delta q_i(t)$ in the non-local lagrangian. It also generates the transformation of the momentum $\mathcal{P}^i(t,\sigma)$ and so, of any functional of the phase space variables. In particular, we will see that, as consistency demands, the hamiltonian (2.5) and the constraints (2.10) and (2.12) are invariant, in the sense that their symmetry transformation vanishes on the hypersurface of phase space determined by the constraints. Let us state a series of results and properties of our gauge generator.

(a) G(t) is a conserved quantity

$$\frac{d}{dt}G(t) = \{G(t), H(t)\} \ + \ \frac{\partial}{\partial t}G(t) \tag{2.18}$$

$$\begin{aligned}
&= \int d\sigma d\sigma' \Bigg[\mathcal{P}^j(t,\sigma)\bigg(\frac{\delta(\delta\mathcal{Q}_j(t,\sigma))}{\delta\mathcal{Q}_i(t,\sigma')}\mathcal{Q}_i{}'(t,\sigma') - \partial_\sigma\delta(\sigma-\sigma')\delta\mathcal{Q}_j(t,\sigma') + \\
&\qquad + \frac{\delta(\delta\mathcal{Q}_j(t,\sigma))}{\delta\Lambda(t,\sigma')}\dot{\Lambda}(t,\sigma')\bigg) - \delta(t,\sigma)\bigg(\frac{\delta\mathcal{K}(t,\sigma)}{\delta\mathcal{Q}_i(t,\sigma')}\mathcal{Q}_i{}'(t,\sigma') - \\
&\qquad - \frac{\delta(\mathcal{L}(t,\sigma))}{\delta\mathcal{Q}_i(t,\sigma')}\delta\mathcal{Q}_i(t,\sigma') + \frac{\delta\mathcal{K}(t,\sigma)}{\delta\Lambda(t,\sigma')}\dot{\Lambda}(t,\sigma')\bigg)\Bigg] = 0\,.
\end{aligned} \tag{2.19}$$

The last term of (2.18) is an explicit t derivative through $\Lambda(t,\sigma)$. In order to show (2.19) we need to use the symmetry condition (2.15) and the condition on $\Lambda(t,\sigma)$ in (2.16).

(b) All the constraints are invariant under the symmetry transformations

Let us show first the invariance of (2.12), which is nothing but the invariance of the equations of motion, as was to expected for $G(t)$ generating a symmetry,

$$\begin{aligned}
\{\tilde{\varphi}^i(t,\sigma), G(t)\} &= \left\{\int d\sigma''\mathcal{E}^i(t,\sigma'',\sigma), \int d\sigma'[\mathcal{P}^j(t,\sigma')\delta\mathcal{Q}_j(t,\sigma') - \delta(\sigma')\mathcal{K}(t,\sigma')]\right\} \\
&= \int d\sigma' d\sigma'' \frac{\delta^2\mathcal{L}(t,\sigma'')}{\delta\mathcal{Q}_j(t,\sigma')\delta\mathcal{Q}_i(t,\sigma)}\delta\mathcal{Q}_j(t,\sigma') \\
&= \int d\sigma' \frac{\delta\tilde{\varphi}^j(t,\sigma')}{\delta\mathcal{Q}_i(t,\sigma)}\delta\mathcal{Q}_j(t,\sigma') \\
&= - \int d\sigma'\tilde{\varphi}^j(t,\sigma')\frac{\delta(\delta\mathcal{Q}_j(t,\sigma'))}{\delta\mathcal{Q}_i(t,\sigma)} \approx 0\,,
\end{aligned} \tag{2.20}$$

where we have used an identity obtained from (2.15),

$$\int d\sigma d\sigma'\mathcal{E}^j(t,\sigma,\sigma')\delta\mathcal{Q}_j(t,\sigma') = \int d\sigma'\tilde{\varphi}^j(t,\sigma')\delta\mathcal{Q}_j(t,\sigma') = 0\,. \tag{2.21}$$

Let us show now the invariance of the other constraint (2.10). Using (2.15) and (2.21),

$$\begin{aligned}
\{\varphi^i(t,\sigma), G(t)\} &= - \int d\sigma'\varphi^j(t,\sigma')\frac{\delta(\delta\mathcal{Q}_j(t,\sigma'))}{\delta\mathcal{Q}_i(t,\sigma)} - \int d\sigma'\bigg[\int d\sigma''\chi(\sigma',-\sigma'') \times \\
&\quad \times\mathcal{E}^j(t;\sigma'',\sigma')\frac{\delta(\delta\mathcal{Q}_j(t,\sigma'))}{\delta\mathcal{Q}_i(t,\sigma)} - \delta(\sigma')\frac{\delta(\mathcal{K}(t,\sigma'))}{\delta\mathcal{Q}_i(t,\sigma)} + \\
&\quad + \int d\sigma''\chi(\sigma,-\sigma'')\frac{\delta\mathcal{E}^i(t;\sigma'',\sigma)}{\delta\mathcal{Q}_j(t,\sigma')}\delta\mathcal{Q}_j(t,\sigma')\bigg] \\
&= - \int d\sigma'\varphi^j(t,\sigma')\frac{\delta(\delta\mathcal{Q}_j(t,\sigma'))}{\delta\mathcal{Q}_i(t,\sigma)} + \int d\sigma'\chi(\sigma,-\sigma')\tilde{\varphi}^j(t,\sigma') \times \\
&\qquad \times \frac{\delta(\delta\mathcal{Q}_j(t,\sigma'))}{\delta\mathcal{Q}_i(t,\sigma)} \approx 0\,.
\end{aligned} \tag{2.22}$$

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Thus we have shown that the constraint surface defined by $\varphi \approx \tilde{\varphi} \approx 0$ is invariant under the transformations generated by $G(t)$.

(c) Our hamiltonian (2.5) is the generator of time translations

Consider a non-local lagrangian in (2.1) that does not depend on t explicitly, so that time translation is a symmetry of the lagrangian. To show that the generator of such a symmetry is our hamiltonian H in (2.5) and that it is conserved, we should simply show that we recover its expression (2.5) from the general form of the generator (2.14). Indeed, the lagrangian changes as $\delta L^{\rm non} = \varepsilon \dot{L}^{\rm non}$ under a time translation $\delta q_i(t) = \varepsilon \dot{q}_i(t)$. The corresponding generator in the present formalism is, using (2.14)

$$G_H(t) = \int d\sigma [\, \mathcal{P}^i(t,\sigma)(\varepsilon \mathcal{Q}_i{}'(t,\sigma)) - \delta(\sigma)(\varepsilon \mathcal{L}(t,\sigma))] \,, \tag{2.23}$$

which is ε times the hamiltonian (2.5). In this case the conservation of the constraints (2.10) and (2.12) is understood also from (2.22) and (2.20). Our hamiltonian in the 1+1 theory being the generator of time translations is telling us that we should consider it as giving the energy of the system. Actually, as we show in the appendix for the U(1) commutative case, if we were working in this $d+1$ formalism but for a *local* theory, we can always use the system of constraints to reduce the redundant extra coordinates and obtain the ordinary hamiltonian of the local theory in d dimensions. Nevertheless, for a truly non local theory, there is no such a simplification and the phase space is infinite dimensional. Our discussion then shows that it is the hamiltonian (2.5) that we should use for computing the energy of the system.

To summarize this chapter, we have constructed the hamiltonian symmetry generators of a general non-local theory working in a $d+1$ dimensional space. In this formulation original gauge symmetries in d dimensions are rigid symmetries in the $d+1$ dimensional space. This way of understanding of gauge symmetries is also useful for ordinary higher derivative theories, see appendix and [12]. The rest of this paper will be mainly devoted to illustrate how our formalism is applied to the case of the non commutative U(1) theory.

3. U(1) non-commutative gauge theory

3.1 Brief review

The magnetic U(1) non-commutative (NC) gauge theory appears in the decoupling limit of D-p branes in the presence of a constant NS-NS two form [5]. The theory could formally be extended to the electric case. However in this case the field theory is acausal [13, 14] and non-unitary [15, 16]. In terms of strings this is because there is an obstruction to the decoupling limit in the case of an electromagnetic background [17]–[21]. Here we are interested in the most general case of *space-time* non-commutativity with $\theta^{0i} \neq 0$.[3]

We consider the U(1) (rank one) *NC* Maxwell theory in d dimensions with the action

$$S = \int d^dx \Big(-\frac{1}{4}\, \widehat{F}_{\mu\nu} \widehat{F}^{\mu\nu} \Big) , \tag{3.1}$$

where $\widehat{F}_{\mu\nu}$ is the field strength of the U(1) *NC* gauge potential $\widehat{A}_\mu$ defined by[4]

$$\widehat{F}_{\mu\nu} = \partial_\mu \widehat{A}_\nu - \partial_\nu \widehat{A}_\mu - i[\widehat{A}_\mu, \widehat{A}_\nu] . \tag{3.2}$$

The commutators in this paper are defined by the Moyal * product as

$$[f,g] \equiv f*g - g*f\,, \qquad f(x)*g(x) = [e^{i(\theta^{\mu\nu}/2)\partial_{\alpha^\mu}\partial_{\beta^\nu}} f(x+\alpha) g(x+\beta)]_{\alpha=\beta=0} . \tag{3.3}$$

The EL equation of motion is

$$\widehat{D}_\mu \widehat{F}^{\mu\nu} = 0 , \tag{3.4}$$

where the covariant derivative is defined by $\widehat{D} = \partial - i[\widehat{A},\ \ ]$.

The gauge transformation is

$$\delta \widehat{A}_\mu = \widehat{D}_\mu \widehat{\lambda} \tag{3.5}$$

and it satisfies a non-abelian gauge algebra,

$$(\delta_{\widehat{\lambda}} \delta_{\widehat{\lambda}'} - \delta_{\widehat{\lambda}'} \delta_{\widehat{\lambda}}) \widehat{A}_\mu = -i \widehat{D}_\mu [\widehat{\lambda}, \widehat{\lambda}'] . \tag{3.6}$$

Since the field strength transforms covariantly as

$$\delta \widehat{F}_{\mu\nu} = -i[\widehat{F}_{\mu\nu}, \widehat{\lambda}] \tag{3.7}$$

the lagrangian density of (3.1) transforms as

$$\delta \Big(-\frac{1}{4}\, \widehat{F}_{\mu\nu} \widehat{F}^{\mu\nu} \Big) = \frac{i}{2}\, [\widehat{F}_{\mu\nu}, \widehat{\lambda}]\ \widehat{F}^{\mu\nu} . \tag{3.8}$$

Using $\int dx (f*g) = \int dx (fg)$ and the associativity of the *star* product (3.8) becomes a total divergence, as was to be expected for (3.5) being a symmetry. So the action (3.1) is invariant under the U(1) *NC* transformations.

3.2 Going to the $d+1$ formalism

The lagrangian (3.1) contains time derivatives of infinite order and is non-local. The *NC* gauge transformation (3.5) is also non-local since, for electric backgrounds ($\theta^{0i} \neq 0$), it contains time derivatives of infinite order in λ. Let us now proceed to construct the hamiltonian and the generator for the U(1) *NC* theory using the formalism introduced in the last section. The canonical structure will be realized in the $d+1$ dimensional formalism. Corresponding to the d dimensional gauge potential $\widehat{A}_\mu(t,\mathbf{x})$, we denote the gauge potential in $d+1$ dimensional one as $\widehat{\mathcal{A}}_\mu(t,\sigma,\mathbf{x})$.[5] We

[3] A hamiltonian formalism for the magnetic theory ($\theta^{0i} = 0$) is analyzed in [22].

[4] We put "hats" on the quantities of the *NC* theory.

[5] From now on we will use calligraphic letters for fields in the $d+1$ formalism.

regard t as the evolution "time". Now $x^0 = \sigma$ is the coordinate denoted by σ of $q_i(t,\sigma)$ in the last section. The other $(d-1)$ spatial coordinates $\mathbf{x}$ correspond to the indices i of $q_i(t,\sigma)$. The signature of $d+1$ space is $(-,-,+,+,\ldots,+)$.

The canonical system equivalent to the non-local action (3.1) is defined by the hamiltonian (2.5) and two constraints, (2.10) and (2.12). For our present theory, the hamiltonian is

$$H(t) = \int d^dx [\widehat{\Pi}^\nu(t,x)\partial_{x^0}\widehat{\mathcal{A}}_\nu(t,x) - \delta(x^0)\mathcal{L}(t,x)]\,, \tag{3.9}$$

where $\widehat{\Pi}^\nu$ is a momentum for $\widehat{\mathcal{A}}_\nu$ and

$$\begin{aligned}\mathcal{L}(t,x) &= -\frac{1}{4}\,\widehat{\mathcal{F}}_{\mu\nu}(t,x)\widehat{\mathcal{F}}^{\mu\nu}(t,x)\,,\\ \widehat{\mathcal{F}}_{\mu\nu}(t,x) &= \partial_\mu\widehat{\mathcal{A}}_\nu(t,x) - \partial_\nu\widehat{\mathcal{A}}_\mu(t,x) - i[\widehat{\mathcal{A}}_\mu(t,x),\widehat{\mathcal{A}}_\nu(t,x)]\,.\end{aligned} \tag{3.10}$$

Note that using (2.6), now the *star* product is defined with respect to $x^\mu = (\sigma,\mathbf{x})$ instead of $x^\mu = (t,\mathbf{x})$ in (3.3). Thus it contains spatial derivatives of infinite order but no time derivative. The same applies for the hamiltonian, it contains no derivative with respect to t, and so it is a good phase-space quantity, a function of the canonical pairs $(\widehat{\mathcal{A}}_\mu(t,x),\widehat{\Pi}^\mu(t,x))$ with Poisson bracket

$$\{\widehat{\mathcal{A}}_\mu(t,x),\widehat{\Pi}^\nu(t,x')\} = \delta_\mu{}^\nu\ \delta^{(d)}(x-x')\,. \tag{3.11}$$

The momentum constraint (2.10) is

$$\begin{aligned}\varphi^\nu(t,x) &= \widehat{\Pi}^\nu(t,x) + \int dy\ \chi(x^0,-y^0)\widehat{\mathcal{F}}^{\mu\nu}(t,y)\widehat{\mathcal{D}}^y_\mu\delta(x-y)\\ &= \widehat{\Pi}^\nu(t,x) + \delta(x^0)\widehat{\mathcal{F}}^{0\nu}(t,x) - \frac{i}{2}\Big(\epsilon(x^0)[\widehat{\mathcal{F}}^{\mu\nu},\widehat{\mathcal{A}}_\mu] - [\epsilon(x^0)\widehat{\mathcal{F}}^{\mu\nu},\widehat{\mathcal{A}}_\mu]\Big)\\ &\approx 0\end{aligned} \tag{3.12}$$

while the constraint (2.11), obtained from the consistency of the above one, turns out to be

$$\tilde{\varphi}^\nu(t,x) = \widehat{\mathcal{D}}_\mu\widehat{\mathcal{F}}^{\mu\nu}(t,x) \approx 0\,. \tag{3.13}$$

Note that these constraints are reducible since $\widehat{\mathcal{D}}_\mu\tilde{\varphi}^\mu \equiv 0$. They reproduce the EL equation of motion (3.4) using the Hamilton equation (2.6),

$$\partial_t\widehat{\mathcal{A}}_\mu(t,x) = \{\widehat{\mathcal{A}}_\mu(t,x),H(t)\} = \partial_{x^0}\widehat{\mathcal{A}}_\mu(t,x) \tag{3.14}$$

and the identification (2.9), $\widehat{\mathcal{A}}_\mu(t,x^\nu) = \widehat{A}_\mu(t+x^0,\mathbf{x})$. Since the lagrangian of (3.1) has translational invariance, the hamiltonian (3.9), as well as the constraints (3.12) and (3.13), are conserved.

To compute the generator of the U(1) *NC* transformation, we apply (2.14) to our case

$$G[\widehat{\Lambda}] = \int dx[\widehat{\Pi}^\mu\delta\widehat{\mathcal{A}}_\mu - \delta(x^0)\mathcal{K}^0]\,, \tag{3.15}$$

where the last term must be evaluated from surface term appearing in the variation of the lagrangian

$$\int dx[-\delta(x^0)\mathcal{K}^0] = \int dx\Big[\frac{\epsilon(x^0)}{2}\partial_\mu\mathcal{K}^\mu\Big] = \int dx\Big[\frac{\epsilon(x^0)}{2}\delta\mathcal{L}\Big]\,. \tag{3.16}$$

Using (3.8) the U(1) generator becomes

$$G[\widehat{\Lambda}] = \int dx\left[\widehat{\Pi}^\mu\widehat{\mathcal{D}}_\mu\widehat{\Lambda} + \frac{i}{4}\epsilon(x^0)\widehat{\mathcal{F}}_{\mu\nu}\,[\widehat{\mathcal{F}}^{\mu\nu},\widehat{\Lambda}]\right], \tag{3.17}$$

where, as discussed in (2.16), $\widehat{\Lambda}(t,x^\mu)$ must be an arbitrary function satisfying

$$\dot{\widehat{\Lambda}}(t,x^\mu) = \partial_{x^0}\widehat{\Lambda}(t,x^\mu)\,. \tag{3.18}$$

The generator can be expressed as a linear combination of the constraints,

$$G[\widehat{\Lambda}] = \int dx\widehat{\Lambda}\left[-(\widehat{\mathcal{D}}_\mu\varphi^\mu) - \delta(x^0)\tilde{\varphi}^0 + \frac{i}{2}\Big(\epsilon(x^0)[\tilde{\varphi}^\nu,\widehat{\mathcal{A}}_\nu] - [\epsilon(x^0)\tilde{\varphi}^\nu\,,\widehat{\mathcal{A}}_\nu]\Big)\right]. \tag{3.19}$$

The fact that the generator (3.19) is a sum of constraints shows explicitly the conservation of the generator on the constraint surface. It also means the U(1) invariance of the hamiltonian on the constraint surface. Furthermore $G[\widehat{\Lambda}]$ is conserved, without using constraints, for $\widehat{\Lambda}(t,x)$ satisfying (3.18),

$$\frac{d}{dt}G[\widehat{\Lambda}] = \{G[\widehat{\Lambda}],H\} \;+\; \frac{\partial}{\partial t}G[\widehat{\Lambda}] = 0 \tag{3.20}$$

in agreement with (2.19).

Finally, the hamiltonian turns out to be

$$H = G[\widehat{\mathcal{A}}_0] + \int dx\varphi^i\widehat{\mathcal{F}}_{0i} + E_L\,, \tag{3.21}$$

where the first term is the U(1) generator (3.19) replacing the parameter $\widehat{\Lambda}$ by $\widehat{\mathcal{A}}_0$. The last term E_L is the only relevant one on the constraint surface, and it is

$$\begin{aligned} E_L = &\int dx\delta(x^0)\{\frac{1}{2}\widehat{\mathcal{F}}^2_{0i} + \frac{1}{4}\widehat{\mathcal{F}}^2_{ij}\} + \frac{i}{2}\int dx\widehat{\mathcal{A}}_0\left(\frac{1}{2}[\widehat{\mathcal{F}}^{ij},\epsilon(x^0)\widehat{\mathcal{F}}_{ij}] - [\widehat{\mathcal{F}}^{0i},\epsilon(x^0)\widehat{\mathcal{F}}_{0i}]\right) + \\ &+\frac{i}{2}\int dx\widehat{\mathcal{A}}_j\left([\widehat{\mathcal{F}}_{0i},\epsilon(x^0)\widehat{\mathcal{F}}^{ij}] - [\epsilon(x^0)\widehat{\mathcal{F}}_{0i},\widehat{\mathcal{F}}^{ij}]\right). \end{aligned} \tag{3.22}$$

This expression is useful, for example, to evaluate the energy of classical configurations of the theory. The first term has the same form as the "energy" of the commutative U(1) theory. The last two terms are non-local contributions. However they vanish in two cases, (1) in $\theta^{0i} = 0$ (magnetic) background and (2) for t independent solutions of $\mathcal{A}_\mu$.

4. Seiberg-Witten map, gauge generators and hamiltonians

Seiberg and Witten [5] have introduced a map between the gauge potential A_μ in a U(1) commutative and $\widehat{A}_\mu$ in an U(1) *NC* theories. Here we show that the Seiberg-Witten (SW) map for the space-time U(1) *NC* theories can be viewed as a *canonical transformation* in the hamiltonian formalism in $d+1$ dimensions. This makes explicit the physical equivalence of both theories. By finding the corresponding generating functional, we are able to map quantities between them. In particular, we show how the gauge generator and the hamiltonian obtained in the previous section for the *NC* case are mapped to those of the commutative theory.

4.1 The d formalism

We recall that the SW map from the U(1) commutative connection A_μ to the U(1) *NC* one $\widehat{A}_\mu$ looks like

$$\widehat{A}_\mu = A_\mu + \frac{1}{2}\theta^{\rho\sigma} A_\sigma (2\partial_\rho A_\mu - \partial_\mu A_\rho) + \cdots . \tag{4.1}$$

In the following discussions we keep terms only up to the first order in θ and higher power terms of θ, indicated by $\ldots$, are omitted.

Under a commutative U(1) transformation of $\delta A_\mu = \partial_\mu \lambda$, the mapped $\widehat{A}_\mu$ transforms as

$$\delta \widehat{A}_\mu = \partial_\mu \Big\{ \lambda + \frac{1}{2}\theta^{\rho\sigma} A_\sigma \partial_\rho \lambda \Big\} + \theta^{\rho\sigma} \partial_\sigma \lambda \partial_\rho A_\mu = \widehat{D}_\mu \widehat{\lambda} \,. \tag{4.2}$$

Note that although the field $\widehat{A}_\mu$ defined above transforms as U(1) *NC* gauge potentials the gauge transformation parameter function $\widehat{\lambda}$ is now gauge field dependent

$$\widehat{\lambda}(\lambda, A) = \lambda + \frac{1}{2}\theta^{\rho\sigma} A_\sigma \partial_\rho \lambda \,. \tag{4.3}$$

The field strength $\widehat{F}_{\mu\nu}$ defined as in (3.2) is, in terms of the commutative fields A_μ and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$, as

$$\widehat{F}_{\mu\nu} = F_{\mu\nu} + \theta^{\rho\sigma} F_{\rho\mu} F_{\sigma\nu} - \theta^{\rho\sigma} A_\rho \partial_\sigma F_{\mu\nu} \tag{4.4}$$

and transforms under $\delta A_\mu = \partial_\mu \lambda$ covariantly as

$$\delta \widehat{F}_{\mu\nu} = -\theta^{\rho\sigma} \partial_\rho \lambda \partial_\sigma F_{\mu\nu} = -i[F_{\mu\nu}, \lambda] = -i[\widehat{F}_{\mu\nu}, \widehat{\lambda}] \,. \tag{4.5}$$

4.2 The $d+1$ formalism

In the $d+1$ dimensional hamiltonian formalism we can regard the mapping (4.1) as a canonical transformation. Denoting the $d+1$ dimensional potentials $\widehat{\mathcal{A}}_\mu(t,x)$ and $\mathcal{A}_\mu(t,x)$ corresponding to d dimensional ones $\widehat{A}_\mu(t,\mathbf{x})$ and $A_\mu(t,\mathbf{x})$ respectively,[6] the generating function turns out to be

$$W(\mathcal{A},\widehat{\Pi}) = \int d^dx\,\widehat{\Pi}^\mu \left(\mathcal{A}_\mu + \frac{1}{2}\theta^{\rho\sigma}\mathcal{A}_\sigma(2\partial_\rho\mathcal{A}_\mu - \partial_\mu\mathcal{A}_\rho)\right) + W^0(\mathcal{A})\,, \tag{4.6}$$

where $W^0(\mathcal{A})$ is any function of $\mathcal{A}_\mu$ of order θ. It generates the transformation of $\mathcal{A}_\mu$ as in (4.1)

$$\widehat{\mathcal{A}}_\mu = \mathcal{A}_\mu + \frac{1}{2}\theta^{\rho\sigma}\mathcal{A}_\sigma(2\partial_\rho\mathcal{A}_\mu - \partial_\mu\mathcal{A}_\rho) \tag{4.7}$$

and determines the relation between Π^μ and $\widehat{\Pi}^\mu$, conjugate momenta of $\mathcal{A}_\mu$ and $\widehat{\mathcal{A}}_\mu$ respectively, to be

$$\Pi^\mu = \widehat{\Pi}^\mu + \frac{1}{2}\widehat{\Pi}^\sigma\theta^{\rho\mu}(2\partial_\rho\mathcal{A}_\sigma - \partial_\sigma\mathcal{A}_\rho) - \partial_\rho(\theta^{\rho\sigma}\mathcal{A}_\sigma\widehat{\Pi}^\mu) + \frac{1}{2}\partial_\rho(\widehat{\Pi}^\rho\theta^{\mu\sigma}\mathcal{A}_\sigma) + \frac{\delta W^0(\mathcal{A})}{\delta\mathcal{A}_\mu}\,. \tag{4.8}$$

It can be inverted, to first order in θ, as

$$\widehat{\Pi}^\mu = \Pi^\mu + \theta^{\mu\rho}\Pi^\sigma\mathcal{F}_{\rho\sigma} + \Pi^\mu\frac{1}{2}\theta^{\rho\sigma}\mathcal{F}_{\rho\sigma} + \theta^{\rho\sigma}\mathcal{A}_\sigma\partial_\rho\Pi^\mu - \frac{1}{2}(\partial_\rho\Pi^\rho)\theta^{\mu\sigma}\mathcal{A}_\sigma - \frac{\delta W^0(\mathcal{A})}{\delta\mathcal{A}_\mu}\,. \tag{4.9}$$

Note that the canonical transformation, (4.7) and (4.9), is independent of the concrete theories we are considering.

In the last section the generator of U(1) *NC* theory was obtained in (3.17) as

$$G[\widehat{\Lambda}] = \int dx\left[\widehat{\Pi}^\mu\widehat{\mathcal{D}}_\mu\widehat{\Lambda} \;+\; \frac{i}{4}\epsilon(x^0)\widehat{\mathcal{F}}_{\mu\nu}\,[\widehat{\mathcal{F}}^{\mu\nu},\widehat{\Lambda}]\right]. \tag{4.10}$$

The last term appeared since the original lagrangian $L^{\rm non}$ changes to a surface term as in (3.8) under the gauge transformation. Now we want to see how this generator transforms under the SW map. It is straightforward to show that, for $W^0(\mathcal{A}) = 0$,

$$\int dx[\widehat{\Pi}^\mu\widehat{\mathcal{D}}_\mu\widehat{\Lambda}(\Lambda,\mathcal{A})] = \int dx[\Pi^\mu\partial_\mu\Lambda]\,, \tag{4.11}$$

where

$$\widehat{\Lambda}(\Lambda,\mathcal{A}) = \Lambda + \frac{1}{2}\theta^{\rho\sigma}\mathcal{A}_\sigma\partial_\rho\Lambda\,, \qquad \dot{\Lambda} = \partial_{x^0}\Lambda\,. \tag{4.12}$$

These results are independent of the specific form of lagrangian for U(1) *NC* and commutative gauge theories. On the other hand the term $\delta(\sigma)\mathcal{K}(t,\sigma)$ appearing in (2.14) does depend on the specific theory we are considering. For the U(1) *NC* theory, (3.1), it is nothing but the lagrangian dependent term in (4.10), which expanding to first order in θ

$$\frac{i}{4}\int dx\epsilon(x^0)\widehat{\mathcal{F}}_{\mu\nu}[\widehat{\mathcal{F}}^{\mu\nu},\widehat{\Lambda}] = \frac{1}{4}\int dx\delta(x^0)\theta^{0i}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\partial_i\Lambda\,. \tag{4.13}$$

[6] Remember, hats for fields in the non-commutative theory, and calligraphic letters for fields in the $d+1$ formalism

In this case the generator of U(1) NC transformations can be mapped to that of commutative one

$$
\begin{aligned}
G[\widehat{\Lambda}(\Lambda,\mathcal{A})] &= \int dx \Big\{ \Pi^0 \partial_0 \Lambda + \Big(\Pi^i + \frac{1}{4}\delta(x^0)\theta^{0i}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\Big)\partial_i \Lambda \Big\} - \int dx \frac{\delta W^0(\mathcal{A})}{\delta \mathcal{A}_\mu}\partial_\mu \Lambda \\
&= \int dx [\ \Pi^\mu \partial_\mu \Lambda\ ]
\end{aligned}
\tag{4.14}
$$

if we choose the canonical transformation with

$$
W^0(\mathcal{A}) = \frac{1}{4}\int dx\ \delta(x^0)\ \theta^{0\mu}\mathcal{A}_\mu \mathcal{F}_{\rho\sigma}\mathcal{F}^{\rho\sigma}\,. \tag{4.15}
$$

The right hand side of (4.14) is the well-known generator of the U(1) commutative theory (see the appendix A).

Now we would like to see what is the form of the U(1) hamiltonian obtained from (3.9) under the SW map, (4.7) and (4.9). The U(1) commutative hamiltonian results to be

$$
H^{(c)} = \int dx [\Pi^\nu(t,x)\mathcal{A}'_\nu(t,x) - \delta(x^0)\mathcal{L}^{(c)}(t,x)]\,, \tag{4.16}
$$

where

$$
\mathcal{L}^{(c)}(t,x) = -\frac{1}{4}\mathcal{F}^{\nu\mu}\mathcal{F}_{\nu\mu} - \frac{1}{2}\mathcal{F}^{\mu\nu}\theta^{\rho\sigma}\mathcal{F}_{\rho\mu}\mathcal{F}_{\sigma\nu} + \frac{1}{8}\theta^{\nu\mu}\mathcal{F}_{\nu\mu}\mathcal{F}_{\rho\sigma}\mathcal{F}^{\rho\sigma}\,. \tag{4.17}
$$

But this is nothing but the $d{+}1$ dimensional hamiltonian that we would have obtained from an abelian U(1) gauge theory with lagrangian

$$
L^{(c)}(t,\mathbf{x}) = -\frac{1}{4}F^{\nu\mu}F_{\nu\mu} - \frac{1}{2}F^{\mu\nu}\theta^{\rho\sigma}F_{\rho\mu}F_{\sigma\nu} + \frac{1}{8}\theta^{\nu\mu}F_{\nu\mu}F_{\rho\sigma}F^{\rho\sigma} \tag{4.18}
$$

in d dimensions. One can check that this lagrangian is, up to a total derivative, the expansion of the Born-Infield action up to order F^3, when written in terms of the open string parameters [5].[7]

$$
L^{(c)} \sim 1 - \sqrt{-\det(\eta_{\mu\nu} - \theta_{\mu\nu} + F_{\mu\nu})} \sim 1 - \sqrt{-\det(\eta_{\mu\nu} + \widehat{F}_{\mu\nu})}\,. \tag{4.19}
$$

5. BRST symmetry

In this section we will conclude our work with the U(1) NC gauge theory by studying its BRST and field-antifield properties. First of all, we will study the BRST symmetry [23, 24] at classical and quantum levels. We will construct the BRST charge and the BRST invariant hamiltonian working with the $d+1$ dimensional formulation, and we will check the nilpotency of the BRST generator. Then, in order to map

[7] We acknowledge discussions with Joan Simón on this point.

the BRST charges and hamiltonians of the U(1) NC and commutative U(1) gauge theories, we will generalize the SW map to the superphase space.

Finally, in the last subsection, we will also study the BRST symmetry at lagrangian level using the field-antifield formalism [25, 26], for a review see [27]–[29]. We will construct the solution of the classical master equation in the classical and gauge fixed basis. We will also realize the SW map as an antibracket canonical transformation.

5.1 Hamiltonian BRST charge

The BRST symmetry at classical level encodes the classical gauge structure through the nilpotency of the BRST transformations of the classical fields and ghosts [30]–[32]. The BRST symmetry of the classical fields is constructed from the gauge transformation by changing the gauge parameters by ghost fields.

Let us consider again the U(1) NC theory still in d dimensions. Its BRST transformations are

$$\begin{aligned} \delta_B \widehat{A}_\mu &= \widehat{D}_\mu \widehat{C}\,, & \qquad \delta_B \widehat{C} &= -i\widehat{C} * \widehat{C}\,, \\ \delta_B \widehat{\overline{C}} &= \widehat{B}\,, & \qquad \delta_B \widehat{B} &= 0\,, \end{aligned} \tag{5.1}$$

where $\widehat{C}, \widehat{\overline{C}}, \widehat{B}$ are the ghost, antighost and auxiliary field respectively.

These are again a symmetry of the lagrangian associated with (3.1), since its change under the BRST transformations is

$$\delta_B L = \frac{i}{2}\,[\widehat{F}_{\mu\nu}, \widehat{C}]\;\widehat{F}^{\mu\nu}\,. \tag{5.2}$$

which, as in (3.8), can be shown a total divergence. We can construct the gauge fixing lagrangian $\widehat{L}_{gf+FP}$ by adding the proper term of the form $\delta_B\widehat{\Psi}$. In this case, the gauge fixing fermion is

$$\widehat{\Psi} = \widehat{\overline{C}}(\partial^\mu \widehat{A}_\mu + \alpha\widehat{B})\,. \tag{5.3}$$

Then the $\widehat{L}_{gf+FP}$ is, up to a total derivative,

$$\widehat{L}_{gf+FP} = -\partial^\mu \widehat{\overline{C}}\;\widehat{D}_\mu \widehat{C} + \widehat{B}(\partial^\mu \widehat{A}_\mu + \alpha\widehat{B})\,. \tag{5.4}$$

By construction, this term does not spoil the symmetry. Indeed

$$\delta_B \widehat{L}_{gf+FP} = \partial^\mu(\widehat{B}\widehat{D}_\mu\widehat{C}). \tag{5.5}$$

In order to construct the generator of the BRST transformations and the BRST invariant hamiltonian we should use the $d+1$ dimensional formulation. We denote the $d+1$ dimensional fields corresponding to the d dimensional ones $\widehat{C}, \widehat{\overline{C}}, \widehat{B}$, using with the calligraphic letters, as $\widehat{\mathcal{C}}, \widehat{\overline{\mathcal{C}}}, \widehat{\mathcal{B}}$ respectively. The results are that the BRST invariant hamiltonian is given by

$$H(t) = H^{(0)} + H^{(1)}\,, \tag{5.6}$$

$$H^{(0)} = \int dx [\widehat{\Pi}^{\nu}(t,x)\widehat{\mathcal{A}}'_{\nu}(t,x) + \widehat{\mathcal{P}_c}(t,x)\widehat{\mathcal{C}}'(t,x) \; - \; \delta(x^0)\widehat{\mathcal{L}}^0(t,x)]\,, \tag{5.7}$$

$$H^{(1)} = \int dx [\widehat{\mathcal{P}}_{\mathcal{B}}\widehat{\mathcal{B}}'(t,x) + \widehat{\mathcal{P}_{\overline{c}}}(t,x)\widehat{\overline{\mathcal{C}}}'(t,x) - \delta(x^0)\widehat{\mathcal{L}}_{gf+FP}(t,x)]\,, \tag{5.8}$$

while the BRST charge is

$$Q_B = Q_B^{(0)} \; + \; Q_B^{(1)}\,, \tag{5.9}$$

$$Q_B^{(0)} = \int dx \left[\widehat{\Pi}^{\mu}\widehat{\mathcal{D}}_{\mu}\widehat{\mathcal{C}} - i\widehat{\mathcal{P}_{\mathcal{C}}} * \widehat{\mathcal{C}} * \widehat{\mathcal{C}} + \frac{1}{2}\epsilon(x^0)\delta_B\widehat{\mathcal{L}}^0(t,x)\right], \tag{5.10}$$

$$Q_B^{(1)} = \int dx \left[\,\widehat{\mathcal{P}_{\overline{\mathcal{C}}}}\widehat{\mathcal{B}} + \frac{1}{2}\epsilon(x^0)\delta_B\widehat{\mathcal{L}}_{gf+FP}(t,x)\right]. \tag{5.11}$$

It is an analogue of the BFV charge [33, 34] for U(1) *NC* theory. $H^{(0)}$, $Q_B^{(0)}$ are the "gauge unfixed" and the H, Q_B are "gauge fixed" hamiltonians and BRST charges.

Using the graded symplectic structure of the superphase space [35]

$$\begin{aligned} \{\widehat{\mathcal{A}}_{\mu}(t,x), \widehat{\Pi}^{\nu}(t,x')\} &= \delta_{\mu}{}^{\nu}\delta^{(d)}(x-x')\,, & \{\widehat{\mathcal{C}}(t,x), \widehat{\mathcal{P}}_{\widehat{\mathcal{C}}}(t,x')\} &= \delta^{(d)}(x-x')\,, \\ \{\widehat{\overline{\mathcal{C}}}(t,x), \widehat{\mathcal{P}}_{\widehat{\overline{\mathcal{C}}}}(t,x')\} &= \delta^{(d)}(x-x')\,, & \{\widehat{\mathcal{B}}(t,x), \widehat{\mathcal{P}}_{\widehat{\mathcal{B}}}(t,x')\} &= \delta^{(d)}(x-x') \end{aligned} \tag{5.12}$$

we have

$$\{H^{(0)}, Q_B^{(0)}\} = \{Q_B^{(0)}, Q_B^{(0)}\} = 0\,, \tag{5.13}$$

and

$$\{H, Q_B\} = \{Q_B, Q_B\} = 0\,. \tag{5.14}$$

Thus the BRST charges are nilpotent and the hamiltonians are BRST invariant both in the gauge unfixed and the gauge fixed levels.

5.2 Seiberg-Witten map in superphase space

Now we would like to see how the BRST charges and the BRST invariant hamiltonians of the *NC* and commutative gauge theories are related. In order to do that we will extend the SW map to a canonical transformation in the superphase space $(\mathcal{A}, \mathcal{C}, \overline{\mathcal{C}}, \mathcal{B}, \Pi, \mathcal{P}_{\mathcal{C}}, \mathcal{P}_{\overline{\mathcal{C}}}, \mathcal{P}_{\mathcal{B}})$. We introduce the generating function

$$\begin{aligned} W(\mathcal{A}, \mathcal{C}, \overline{\mathcal{C}}, \mathcal{B}, \widehat{\Pi}, \widehat{\mathcal{P}}_{\mathcal{C}}, \widehat{\mathcal{P}}_{\overline{\mathcal{C}}}, \widehat{\mathcal{P}}_{\mathcal{B}}) = \int dx \Big[\widehat{\Pi}^{\mu}\left(\mathcal{A}_{\mu} + \frac{1}{2}\theta^{\rho\sigma}\mathcal{A}_{\sigma}(2\partial_{\rho}\mathcal{A}_{\mu} - \partial_{\mu}\mathcal{A}_{\rho})\right) + \\ + \widehat{\mathcal{P}}_{\mathcal{C}}\left(\mathcal{C} + \frac{1}{2}\theta^{\rho\sigma}\mathcal{A}_{\sigma}\partial_{\rho}\mathcal{C}\right) + \widehat{\mathcal{P}}_{\overline{\mathcal{C}}}\overline{\mathcal{C}} + \widehat{\mathcal{P}}_{\mathcal{B}}\mathcal{B}\Big] + \\ + W^0(\mathcal{A}, \mathcal{C}) \; + \; W^1(\mathcal{A}, \mathcal{C}, \overline{\mathcal{C}}, \mathcal{B})\,, \end{aligned} \tag{5.15}$$

where $W^0(\mathcal{A},\mathcal{C})$ depends on the specific form of the U(1) *NC* lagrangian and $W^1(\mathcal{A},\mathcal{C},\overline{\mathcal{C}},B)$ also on the form of the gauge fixing. For the U(1) *NC* theory and for the gauge fixing (5.3), we have

$$W^0(\mathcal{A},\mathcal{C}) = \frac{1}{4}\int dx\ \delta(x^0)\ \theta^{0\mu}\mathcal{A}_\mu\mathcal{F}_{\rho\sigma}\mathcal{F}^{\rho\sigma} \tag{5.16}$$

as in (4.15) and

$$\begin{aligned} W^1 = \int dx\frac{1}{2}\epsilon(x^0)\Big[\partial^\mu\Big\{\frac{1}{2}\theta^{\rho\sigma}\mathcal{A}_\sigma(2\partial_\rho\mathcal{A}_\mu - \partial_\mu\mathcal{A}_\rho)\Big\}\mathcal{B}+ \\ + \Big\{\frac{1}{2}\theta^{\rho\sigma}\mathcal{A}_\sigma(2\partial_\rho\mathcal{A}_\mu - \partial_\mu\mathcal{A}_\rho)\partial_\sigma\mathcal{C} + \frac{1}{2}\theta^{\rho\sigma}\mathcal{A}_\sigma\partial_\mu\partial_\rho\mathcal{C}\Big\}\partial^\mu\overline{\mathcal{C}}\ \Big]. \end{aligned} \tag{5.17}$$

The transformations are obtained from the generating function by

$$\widehat{\Phi}^A = \frac{\partial_\ell W}{\partial\widehat{P}_A}, \qquad P_A = \frac{\partial_r W}{\partial\Phi^A}, \tag{5.18}$$

where Φ^A represent any fields, P_A their conjugate momenta, and ∂_r and ∂_ℓ are right and left derivatives respectively.

Explicitly we have

$$\widehat{\mathcal{A}}_\mu = \mathcal{A}_\mu + \frac{1}{2}\theta^{\rho\sigma}\mathcal{A}_\sigma(2\partial_\rho\mathcal{A}_\mu - \partial_\mu\mathcal{A}_\rho)\,, \tag{5.19}$$

$$\widehat{\mathcal{C}} = \mathcal{C} + \frac{1}{2}\theta^{\rho\sigma}\mathcal{A}_\sigma\partial_\rho\mathcal{C}\,, \tag{5.20}$$

$$\widehat{\overline{\mathcal{C}}} = \overline{\mathcal{C}}\,, \tag{5.21}$$

$$\widehat{\mathcal{B}} = \mathcal{B}\,, \tag{5.22}$$

and

$$\begin{aligned} \widehat{\Pi}^\mu &= \Pi^\mu + \theta^{\mu\rho}\Pi^\sigma\mathcal{F}_{\rho\sigma} + \Pi^\mu\frac{1}{2}\theta^{\rho\sigma}\mathcal{F}_{\rho\sigma} + \theta^{\rho\sigma}\mathcal{A}_\sigma\partial_\rho\Pi^\mu - \frac{1}{2}(\partial_\rho\Pi^\rho)\theta^{\mu\sigma}\mathcal{A}_\sigma + \\ &\quad + \frac{1}{2}\mathcal{P}_\mathcal{C}\theta^{\mu\sigma}\partial_\sigma\mathcal{C} - \frac{\delta(W^0+W^1)}{\delta\mathcal{A}_\mu}\,, \\ \widehat{\mathcal{P}}_\mathcal{C} &= \mathcal{P}_\mathcal{C} + \frac{1}{2}\theta^{\rho\sigma}\partial_\rho(\mathcal{P}_\mathcal{C}\mathcal{A}_\sigma) - \frac{\delta_r(W^0+W^1)}{\delta\mathcal{C}}\,, \\ \widehat{\mathcal{P}}_{\overline{\mathcal{C}}} &= \mathcal{P}_{\overline{\mathcal{C}}} - \frac{\delta_r W^1}{\delta\overline{\mathcal{C}}}\,, \\ \widehat{\mathcal{P}}_\mathcal{B} &= \mathcal{P}_\mathcal{B} - \frac{\delta_r W^1}{\delta\mathcal{B}}\,. \end{aligned} \tag{5.23}$$

Using this transformation we can rewrite the BRST charge (5.9) as

$$\begin{aligned} Q_B &= Q_B^{(0)} + Q_B^{(1)} \\ &= \int dx[\Pi^\mu\partial_\mu\mathcal{C}\ +\ \mathcal{P}_{\overline{\mathcal{C}}}\mathcal{B} - \delta(x^0)\mathcal{B}\partial^0\mathcal{C}\ ] \\ &= \int dx\Big[\Pi^\mu\partial_\mu\mathcal{C} + \mathcal{P}_{\overline{\mathcal{C}}}\mathcal{B} + \frac{1}{2}\epsilon(x^0)\delta_B\mathcal{L}_{gf+FP}(t,x)\Big]\,, \end{aligned} \tag{5.24}$$

where $\mathcal{L}_{gf+FP}(t,x)$ is the abelian gauge fixing lagrangian and is given by

$$\mathcal{L}_{gf+FP} = -\partial^\mu \overline{\mathcal{C}}\ \partial_\mu \mathcal{C} + \mathcal{B}\ (\partial^\mu \mathcal{A}_\mu + \alpha \mathcal{B})\,. \tag{5.25}$$

The total U(1) hamiltonian (5.6) becomes

$$H = \int dx [\Pi^\nu \mathcal{A}'_\nu + \mathcal{P}_\mathcal{C} \mathcal{C}' + \mathcal{P}_{\overline{\mathcal{C}}} \overline{\mathcal{C}}' + P_\mathcal{B} \mathcal{B}' - \delta(x^0)(\mathcal{L}^{(c)} + \mathcal{L}_{gf+FP})]\,. \tag{5.26}$$

Remember $\mathcal{L}^{(c)}$ is the U(1) commutative lagrangian given in (4.17). Summarizing, we have been successful in mapping the NC and commutative charges in the $d+1$ formalism by generalizing the SW map to a canonical transformation in the superphase space.

5.3 Field-antifield formalism for U(1) non-commutative theory

The field-antifield formalism allows us to study the BRST symmetry of a general gauge theory by introducing a canonical structure at a lagrangian level [25]–[28]. The classical master equation in the classical basis encodes the gauge structure of the generic gauge theory [31, 32]. The solution of the classical master equation in the gauge fixed basis gives the "quantum action" to be used in the path integral quantization. Any two solutions of the classical master equations are related by a canonical transformation in the antibracket sense [36].

Here we will apply these ideas to the U(1) NC theory. Since we work at a lagrangian level we will work in d dimensions. In the classical basis the set of fields and antifields are

$$\Phi^A = \{\widehat{A}_\mu, \widehat{C}\}\,, \qquad \Phi^*_A = \{\widehat{A}^*_\mu, \widehat{C}^*\}\,. \tag{5.27}$$

The solution of the classical master equation

$$(S,S) = 0\,, \tag{5.28}$$

is given by[8]

$$S[\Phi,\Phi^*] = I[\widehat{A}] + \widehat{A}^*_\mu \widehat{D}^\mu \widehat{C} - i\widehat{C}^*(\widehat{C} * \widehat{C})\,, \tag{5.29}$$

where $I[\widehat{A}]$ is the classical action and the antibracket $(\ ,\ )$ is defined by

$$(X,Y) = \frac{\partial_r X}{\partial \Phi^A}\frac{\partial_l Y}{\partial \Phi^*_A} - \frac{\partial_r X}{\partial \Phi^*_A}\frac{\partial_l Y}{\partial \Phi^A}\,. \tag{5.30}$$

The gauge fixed basis can be analyzed by introducing the antighost and auxiliary fields and the corresponding antifields. It can be obtained from the classical basis by considering a canonical transformation, in the antibracket sense,

$$\begin{aligned} \Phi^A &\longrightarrow \Phi^A \\ \Phi^*_A &\longrightarrow \Phi^*_A + \frac{\partial_r \Psi}{\partial \Phi^A} \end{aligned} \tag{5.31}$$

[8] As in usual convention in the antifield formalism, d dimensional integration is understood in summations.

generated by

$$\widehat{\Psi} = \widehat{\overline{C}}\,(\partial^\mu \widehat{A}_\mu + \alpha \widehat{B})\,, \tag{5.32}$$

where $\widehat{\overline{C}}$ is the antighost and $\widehat{B}$ is the auxiliary field. We have

$$S[\Phi, \Phi^*] = \widehat{I}_\Psi + \widehat{A}^{*\mu} \widehat{D}_\mu \widehat{C} - i\widehat{C}^*(\widehat{C} * \widehat{C}) + \widehat{\overline{C}}^*\, \widehat{B}\,, \tag{5.33}$$

where $\widehat{I}_\Psi$ is the "quantum action" and is given by

$$\widehat{I}_\Psi = I[\widehat{A}] + (-\partial_\mu \widehat{\overline{C}}\ \widehat{D}^\mu \widehat{C} + \widehat{B}\ \partial_\mu \widehat{A}^\mu + \alpha \widehat{B}^2)\,. \tag{5.34}$$

The action $\widehat{I}_\Psi$ has well defined propagators and is the starting point of the Feynman perturbative calculations.

Now we would like to study what is the SW map in the space of fields and antifields. We first consider it in the classical basis. In order to do that we construct a canonical transformation in the antibracket sense

$$\widehat{\Phi}^A = \frac{\partial_l F_{cl}[\Phi, \widehat{\Phi}^*]}{\partial \widehat{\Phi}^*_A}\,, \qquad \Phi^*_A = \frac{\partial_r F_{cl}[\Phi, \widehat{\Phi}^*]}{\partial \Phi^A}\,, \tag{5.35}$$

where

$$F_{cl} = \widehat{A}^{*\mu}\left(A_\mu + \frac{1}{2}\theta^{\rho\sigma} A_\sigma(2\partial_\rho A_\mu - \partial_\mu A_\rho)\right) + \widehat{C}^*\left(C + \frac{1}{2}\theta^{\rho\sigma} A_\sigma \partial_\rho C\right). \tag{5.36}$$

The gauge structures of NC and commutative are mapped to each other

$$\widehat{A}^*_\mu \widehat{D}^\mu \widehat{C} - i\widehat{C}^*(\widehat{C} * \widehat{C}) = A^*_\mu \partial^\mu C\,. \tag{5.37}$$

We can generalize the previous results to the gauge fixed basis. In this case the transformations of the antighost and the auxiliary field sectors should be taken into account. The generator of the canonical transformation is modified from (5.36) to

$$F_{gf} = F_{cl} + \left(\widehat{\overline{C}}^* + \frac{1}{2}\theta^{\rho\sigma}\partial^\mu\left(A_\sigma(2\partial_\rho A_\mu - \partial_\mu A_\rho)\right)\right)\overline{C} + \widehat{B}^* B\,. \tag{5.38}$$

Note that the additional term gives rise to new terms in $A^{*\mu}$ and $\overline{C}^*$ while the others remain the same as in the classical basis. In particular

$$\widehat{\overline{C}} = \overline{C}\,, \qquad \widehat{B} = B\,. \tag{5.39}$$

Using the transformation we can express (5.33) and (5.34) as

$$S[\Phi, \Phi^*] = I_\Psi + A^{*\mu}\partial_\mu C + \overline{C}^* B\,, \tag{5.40}$$

where

$$I_\Psi = I[\widehat{A}(A)] + (-\partial_\mu \overline{C}\ \partial^\mu C + B\ \partial_\mu A^\mu + \alpha B^2) \tag{5.41}$$

and $I[\widehat{A}(A)]$ is the classical action in terms of A_μ. This is indeed a quantum action for the commutative U(1) BRST invariant action in the gauge fixed basis. In this way the canonical transformation (5.38) maps the U(1) NC structure of the $S[\Phi, \Phi^*]$ into the commutative one in the gauge fixed basis.

6. Discussions

In this paper the hamiltonian formalism of the non-local theories is discussed by using $d+1$ dimensional formulation [1, 2]. For a given non-local lagrangian in d dimensions the hamiltonian is introduced by (2.5) on the phase space of the $d+1$ dimensional fields. The equivalence with the original non-local theory is assured by imposing two constraints (2.10) and (2.12) consistent with the time evolution. The degrees of freedom of the extra dimension (denoted by coordinate σ) has its origin in the infinite degrees of freedom associated with the non-locality. The fact that we have been led to a theory with "two times" should be intimately related to their acausality [13, 14] and non-unitarity [15, 16].

The $d+1$ formalism is also applicable to *local* and higher derivative theories. In these cases the set of constraints are used to reduce the redundant degrees of freedom of the infinite dimensional phase space, reproducing the standard d dimensional formulations [12].

We have analyzed the symmetry generators of non-local theories in the hamiltonian formalism. As an example we have considered the space-time U(1) *NC* gauge theory. The gauge transformations in d dimensions are described as a rigid symmetry in $d+1$ dimensions. The generators of *rigid* transformations in $d+1$ dimensions turn out to be the generators of *gauge* transformations when the reduction to d dimensions can be performed as is shown for the U(1) commutative gauge theory in the appendix A.

We have extended the Seiberg-Witten map to a canonical transformation. This allows us to map the hamiltonians and the gauge generators of non-commutative and commutative theories. We have also seen explicitly the map of the U(1) *NC* and the BI actions up to F^3. The reason why we were able to discuss the SW map as a canonical transformation is that we have considered the phase space of the commutative theory also in the $d+1$ dimensions.

The BRST symmetry has been analyzed at hamiltonian and lagrangian levels. The relation between the U(1) commutative and *NC* parameter functions is understood as a canonical transformation of the ghosts in the super phase space of the SW map. Using the field-antifield formalism we have seen how the solution of the classical master equation for non-commutative and commutative theories are related by a canonical transformation in the antibracket sense. This results shows that the antibracket cohomology classes of both theories coincide in the space of non-local functionals. The explicit forms of the antibracket canonical transformations could be useful to study the observables, anomalies, etc. in the U(1) *NC* theory.

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A. U(1) commutative Maxwell theory in $d+1$ dimensions

Our $d+1$ formalism can also be used for describing ordinary local theories. As an example of this, we will show how the U(1) commutative Maxwell theory is formulated using the $d+1$ dimensional canonical formalism developed for non-local theories in section 2 and see how it is reduced to the standard canonical formalism in d dimensions.

The canonical $d+1$ system is defined by the hamiltonian (2.5) and two constraints, (2.10) and (2.11). The hamiltonian is

$$H = \int d^d x [\Pi^\nu(t,x)\partial_{x^0}\mathcal{A}_\nu(t,x) - \delta(x^0)\mathcal{L}(t,x)]\,, \tag{A.1}$$

where

$$\begin{aligned}\mathcal{L}(t,x) &= -\frac{1}{4}\,\mathcal{F}_{\mu\nu}(t,x)\mathcal{F}^{\mu\nu}(t,x)\,,\\ \mathcal{F}_{\mu\nu}(t,x) &= \partial_\mu\mathcal{A}_\nu(t,x) - \partial_\nu\mathcal{A}_\mu(t,x)\,.\end{aligned} \tag{A.2}$$

The momentum constraint (2.10) is

$$\begin{aligned}\varphi^\nu(t,x) &= \Pi^\nu(t,x) + \int dy\ \chi(x^0,-y^0)\mathcal{F}^{\mu\nu}(t,y)\partial^y_\mu\delta(x-y)\\ &= \Pi^\nu(t,x) + \delta(x^0)\mathcal{F}^{0\nu}(t,x) \approx 0\end{aligned} \tag{A.3}$$

and the constraint (2.11) is

$$\tilde{\varphi}^\nu(t,x) = \partial_\mu\mathcal{F}^{\mu\nu}(t,x) \approx 0\,. \tag{A.4}$$

The generator of the U(1) transformation is given, using (2.14), by

$$G[\Lambda] = \int dx[\Pi^\mu\partial_\mu\Lambda]\,. \tag{A.5}$$

It is expressed as a linear combination of the constraints,

$$G[\Lambda] = \int dx\Lambda\left[-(\partial_\mu\varphi^\mu) - \delta(x^0)\tilde{\varphi}^0\right]. \tag{A.6}$$

The hamiltonian is expressed using the constraints and the U(1) generator as

$$H = G[\mathcal{A}_0] + \int dx\varphi^i\mathcal{F}_{0i} + \int dx\delta(x^0)\Big\{\frac{1}{2}\mathcal{F}_{0i}{}^2 + \frac{1}{4}\mathcal{F}_{ij}{}^2\Big\}\,. \tag{A.7}$$

The hamiltonian (A.7) as well as the constraints (A.3) and (A.4) contain no time (t) derivative and are functions of the canonical pairs $(\mathcal{A}_\mu(t,x), \Pi^\mu(t,x))$. They are conserved since the Maxwell lagrangian in d dimensions has time translation invariance. The U(1) generator is also conserved, without using constraints, for $\Lambda(t,x)$ satisfying (2.16),

$$\frac{d}{dt}G[\Lambda] = \{G[\Lambda], H\} + \frac{\partial}{\partial t}G[\Lambda] = 0\,, \qquad \dot{\Lambda} = \partial_{x^0}\Lambda\,. \tag{A.8}$$

in agreement with (2.19). Since the parameter Λ is subject to the last relation in (A.8) the U(1) transformations in the $d+1$ dimensional canonical formulation are not gauge but rigid ones. We will see how the gauge transformations appear when it is written in a d dimensional form.

In cases where our lagrangians are local or higher derivative ones it is often convenient to make expansion of the canonical variables using the Taylor basis[37] in reducing them to d dimensional forms. We expand the canonical variables as

$$\mathcal{A}_\mu(t,x) \equiv \sum_{m=0}^{\infty} e_m(x^0)\; A_\mu^{(m)}(t,\mathbf{x})\,, \qquad \Pi^\mu(t,x) \equiv \sum_{m=0}^{\infty} e^m(x^0)\; \Pi^\mu_{(m)}(t,\mathbf{x})\,, \tag{A.9}$$

where $e^\ell(x^0)$ and $e_\ell(x^0)$ are orthonormal basis

$$\begin{aligned} e^\ell(x^0) &= (-\partial_{x^0})^\ell \delta(x^0)\,, & e_\ell(x^0) &= \frac{(x^0)^\ell}{\ell!}\,, \\ \textstyle\int dx^0 e^\ell(x^0)\; e_m(x^0) &= \delta^\ell_m\,, & \textstyle\sum_{\ell=0}^\infty e^\ell(x^0)\; e_\ell(x^{0\prime}) &= \delta(x^0 - x^{0\prime})\,. \end{aligned} \tag{A.10}$$

The $(A_\mu^{(m)}(t,\mathbf{x}), \Pi^\mu_{(m)}(t,\mathbf{x}))$ are d dimensional fields and are the new symplectic coordinates

$$\Omega(t) = \int dx \delta\Pi^\mu(t,x) \wedge \delta\mathcal{A}_\mu(t,x) = \sum_{m=0}^{\infty}\int d\mathbf{x}\delta\Pi^\mu_{(m)}(t,\mathbf{x}) \wedge \delta A_\mu^{(m)}(t,\mathbf{x})\,. \tag{A.11}$$

In terms of them the constraint (A.3) is expressed as

$$\varphi^\mu(t,x) = \sum_{m=0}^{\infty} e^m(x^0)\varphi^\mu_{(m)}(t,\mathbf{x})\,, \tag{A.12}$$

$$\varphi^0_{(m)}(t,\mathbf{x}) = \Pi^0_{(m)}(t,\mathbf{x}) = 0\,, \qquad (m \geq 0)\,, \tag{A.13}$$

$$\varphi^i_{(0)}(t,\mathbf{x}) = \Pi^i_{(0)}(t,\mathbf{x}) - (\mathcal{A}_i^{(1)}(t,\mathbf{x}) - \partial_i\mathcal{A}_0^{(0)}(t,\mathbf{x})) = 0\,, \tag{A.14}$$

$$\varphi^i_{(m)}(t,\mathbf{x}) = \Pi^i_{(m)}(t,\mathbf{x}) = 0\,, \qquad (m \geq 1)\,, \tag{A.15}$$

while the constraint (A.4) is

$$\tilde{\varphi}^\mu(t,x) = \sum_{m=0}^{\infty} e_m(x^0)\tilde{\varphi}^{\mu(m)}(t,\mathbf{x})\,, \tag{A.16}$$

$$\tilde{\varphi}^{i(m)}(t,\mathbf{x}) = \partial_j(\partial_j\mathcal{A}_i^{(m)}(t,\mathbf{x}) - \partial_i\mathcal{A}_j^{(m)}(t,\mathbf{x})) - (\mathcal{A}_i^{(m+2)}(t,\mathbf{x}) - \partial_i\mathcal{A}_0^{(m+1)}(t,\mathbf{x})) = 0\,,$$
$$(m \geq 0)\,, \tag{A.17}$$
$$\tilde{\varphi}^{0(m)}(t,\mathbf{x}) = \partial_i(\mathcal{A}_i^{(m+1)}(t,\mathbf{x}) - \partial_i\mathcal{A}_0^{(m)}(t,\mathbf{x})) = 0\,,$$
$$(m \geq 0)\,. \tag{A.18}$$

It must be noted the identities

$$\tilde{\varphi}^{0(m+1)}(t,\mathbf{x}) = \partial_i\tilde{\varphi}^{i(m)}(t,\mathbf{x})\,, \qquad (m \geq 0)\,. \tag{A.19}$$

Thus the only independent constraint of (A.18) is $m = 0$ case. It can be expressed, using (A.14), as the gauss law constraint,

$$\tilde{\varphi}^{0(0)}(t,\mathbf{x}) = \partial_i\Pi^i_{(0)}(t,\mathbf{x}) = 0\,. \tag{A.20}$$

Following the Dirac's standard procedure of constraints [6] we classify them and eliminate the second class constraints. The constraints (A.15) ($m \geq 2$) are paired with the constraints (A.17) ($m \geq 0$) to form second class sets. They are used to eliminate canonical pairs $(\mathcal{A}_i^{(m)}(t,\mathbf{x}), \Pi^i_{(m)}(t,\mathbf{x})), (m \geq 2)$ as

$$\mathcal{A}_i^{(m)}(t,\mathbf{x}) = \partial_j(\partial_j\mathcal{A}_i^{(m-2)}(t,\mathbf{x}) - \partial_i\mathcal{A}_j^{(m-2)}(t,\mathbf{x})) + \partial_i\mathcal{A}_0^{(m-1)}(t,\mathbf{x})\,,$$
$$\Pi^i_{(m)}(t,\mathbf{x})) = 0\,, \qquad (m \geq 2)\,. \tag{A.21}$$

The constraints (A.15) ($m = 1$) and (A.14) are paired to a second class set and are used to eliminate $(\mathcal{A}_i^{(1)}(t,\mathbf{x}), \Pi^i_{(1)}(t,\mathbf{x}))$ as

$$\mathcal{A}_i^{(1)}(t,\mathbf{x}) = \Pi^i_{(0)}(t,\mathbf{x}) + \partial_i\mathcal{A}_0^{(0)}(t,\mathbf{x})\,,$$
$$\Pi^i_{(1)}(t,\mathbf{x}) = 0\,. \tag{A.22}$$

After eliminating the canonical pairs $(\mathcal{A}_i^{(m)}(t,\mathbf{x}), \Pi^i_{(m)}(t,\mathbf{x})), (m \geq 1)$ using the second class constraints the system is described in terms of the canonical pairs $(\mathcal{A}_i^{(0)}(t,\mathbf{x}), \Pi^i_{(0)}(t,\mathbf{x}))$ and $(\mathcal{A}_0^{(m)}(t,\mathbf{x}), \Pi^0_{(m)}(t,\mathbf{x})), (m \geq 0)$. The Dirac brackets among them remain same as the Poisson brackets. Remember the d dimensional fields are identified by (2.9) as

$$A_\mu(t,\mathbf{x}) = \mathcal{A}_\mu(t,0,\mathbf{x}) = \mathcal{A}_\mu^{(0)}(t,\mathbf{x})\,, \qquad \Pi^\mu(t,\mathbf{x}) = \Pi^\mu_{(0)}(t,\mathbf{x})\,. \tag{A.23}$$

The remaining constraints are (A.20) and (A.13),

$$\partial_i\Pi^i_{(0)}(t,\mathbf{x}) = 0\,, \qquad \Pi^0_{(m)}(t,\mathbf{x}) = 0\,, \qquad (m \geq 0)\,. \tag{A.24}$$

They are first class constraints. The hamiltonian (A.7) in the reduced variables is

$$H(t) = \int d\mathbf{x}\left[\sum_{m=0}^{\infty}\mathcal{A}_0^{(m+1)}(t,\mathbf{x})\Pi^0_{(m)}(t,\mathbf{x}) - \mathcal{A}_0^{(0)}(t,\mathbf{x})(\partial_i\Pi^i_{(0)}(t,\mathbf{x})) + \right.$$
$$\left. + \frac{1}{2}(\Pi^i_{(0)}(t,\mathbf{x}))^2 + \frac{1}{4}(\partial_j\mathcal{A}_i^{(0)}(t,\mathbf{x}) - \partial_i\mathcal{A}_j^{(0)}(t,\mathbf{x}))^2\right]\,. \tag{A.25}$$

The U(1) generator (A.6) is

$$G[\Lambda] = \int d\mathbf{x} \left[\sum_{m=0}^{\infty} \Lambda^{(m+1)}(t,\mathbf{x}) \mathit{\Pi}^0_{(m)}(t,\mathbf{x}) - \Lambda^{(0)}(t,\mathbf{x})(\partial_i \mathit{\Pi}^i_{(0)}(t,\mathbf{x})) \right], \quad (A.26)$$

where

$$\Lambda(t,\lambda) = \sum_{m=0}^{\infty} \Lambda^{(m)}(t,\mathbf{x}) e_m(x^0), \qquad \text{and} \qquad \dot{\Lambda}^{(m)}(t,\mathbf{x}) = \Lambda^{(m+1)}(t,\mathbf{x}). \quad (A.27)$$

The first class constraints $\mathit{\Pi}^0_{(m)}(t,\mathbf{x}) = 0, (m \geq 0)$ in (A.24) mean that $\mathcal{A}_0^{(m)}(t,\mathbf{x})$, $(m \geq 0)$ are the gauge degrees of freedom and we can assign to them any function of $\mathbf{x}$ for all values of m at given time $t = t_0$. It is equivalent to saying that we can assign any function of time to $\mathcal{A}_0^{(0)}(t,\mathbf{x})$ for all value of t, due to the equation of motion $\dot{\mathcal{A}}_0^{(m)}(t,\mathbf{x}) = \mathcal{A}_0^{(m+1)}(t,\mathbf{x})$. In this way we can understand that the hamiltonian (A.25) is equivalent to the standard form of the canonical hamiltonian of the Maxwell theory,

$$\begin{aligned} H(t) = \int d\mathbf{x} \Big[& \dot{A}_0(t,\mathbf{x}) \Pi^0(t,\mathbf{x}) - A_0(t,\mathbf{x})(\partial_i \Pi^i(t,\mathbf{x})) + \\ & + \frac{1}{2}(\Pi^i(t,\mathbf{x}))^2 + \frac{1}{4}(\partial_j A_i(t,\mathbf{x}) - \partial_i A_j(t,\mathbf{x}))^2 \Big] \end{aligned} \quad (A.28)$$

in which $A_0(t,\mathbf{x})$ is arbitrary function of time. In the same manner the U(1) generator (A.26) is

$$G[\Lambda] = \int d\mathbf{x} \left[\dot{\lambda}(t,\mathbf{x}) \Pi^0(t,\mathbf{x}) - \lambda(t,\mathbf{x})(\partial_i \Pi^i(t,\mathbf{x})) \right], \quad (A.29)$$

in which the gauge parameter function $\lambda(t,\mathbf{x}) \equiv \Lambda^{(0)}(t,\mathbf{x})$ is regarded as any function of time.

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